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PERELMAN'S λ-FUNCTIONAL AND THE SEIBERG-WITTEN EQUATIONS

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ABSTRACT. In this paper we study the supremum of Perelman's λ-functional $\lambda_M(g)$ on Riemannian 4-manifold M by using the Seiberg-Witten equations. We prove among others that, for a compact Kähler-Einstein complex surface (M, J, g_0) with negative scalar curvature, (i) If g_1 is a Riemannian metric on M with $\lambda_M(g_1) = \lambda_M(g_0)$, then $\mathrm{Vol}_{g_1}(M) \geq \mathrm{Vol}_{g_0}(M)$. Moreover, the equality holds if and only if g_1 is also a Kähler-Einstein metric with negative scalar curvature. (ii) If $\{g_t\}$, $t \in [-1, 1]$, is a family of Einstein metrics on M with initial metric g_0, then g_t is a Kähler-Einstein metric with negative scalar curvature.

1. INTRODUCTION

In his celebrated paper [H] R.Hamilton introduced the Ricci-flow evolution equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) \tag{1.1}$$

with initial metric $g(0) = g$. The Ricci flow is now a fundamental tool to solve the famous Poincaré conjecture and Thurston's Geometrizaion conjecture, by the works of G. Perelman [Pe1][Pe2]. A fundamental new discovery of Perelman is to prove the Ricci-flow evolution equation is the gradient flow of a so called *Perelman's λ-functional* of a Riemannian manifold(cf. [Pe1][KL]), which may be described as follows: for a smooth function $f \in C^\infty(M)$ on a Riemannian n-manifold with a Riemannian metric g, let

$$\mathcal{F}(g, f) = \int_M (R_g + |\nabla f|^2) e^{-f} dvol_g, \tag{1.2}$$

where R_g is the scalar curvature of g. The Perelman's λ-functional is defined by

$$\lambda_M(g) = \inf_f \{\mathcal{F}(g, f) | \int_M e^{-f} dvol_g = 1\}. \tag{1.3}$$

Note that $\lambda_M(g)$ is the lowest eigenvalue of the operator $-4\triangle + R_g$. Let

$$\overline{\lambda}_M(g) = \lambda_M(g)\mathrm{Vol}_g(M)^{\frac{2}{n}} \tag{1.4}$$

which is invariant up to rescale the metric. Perelman [Pe1] has established the monotonicity property of $\overline{\lambda}_M(g_t)$ along the Ricci flow g_t, namely, the function is non-decreasing along the Ricci flow g_t whenever $\overline{\lambda}_M(g_t) \leq 0$. Therefore, it is interesting to

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study the upper bound of $\overline{\lambda}_M(g)$. This leads to define a diffeomorphism invariant $\overline{\lambda}_M$ of M due to Perelman (cf. [Pe2] [KL]) by

$$\overline{\lambda}_M = \sup_{g\in\mathcal{M}} \overline{\lambda}_M(g), \tag{1.5}$$

where $\mathcal{M}$ is the set of Riemannian metrics on M. It is easy to see that $\overline{\lambda}_M = 0$ if M admits a volume collapsing with bounded scalar curvature but does not admit any metric with positive scalar curvature (cf. [KL]). By a deep result of Perelman (cf. [Pe2][KL]), for a 3-manifold M which does not admit a metric of positive scalar curvature, $(-\overline{\lambda}_M)^{\frac{3}{2}}$ is proportional to the minimal volume of the manifold.

The invariant $\overline{\lambda}_M$ may take value $+\infty$, e.g., $M = S^2 \times S^2$. Thus, it seems only interesting when $\overline{\lambda}_M \leq 0$, i.e, when M does not admit any metric of positive scalar curvature.

In this paper we will investigate $\overline{\lambda}_M$ by using the Seiberg-Witten monopole equations for a 4-manifold M. We say a Spin^c-structure (or equivalently its first Chern class) is a *monopole class* if the Seiberg-Witten monopople equations has an irreducible solution.

Our first result is as follows:

Theorem 1.1. *Let $(M, \mathfrak{c})$ be a smooth compact closed oriented 4-manifold with a* Spin^c-*structure $\mathfrak{c}$. If the first Chern class c_1 of $\mathfrak{c}$ is a monopole class of M satisfying that $c_1^2[M] > 0$. Then, for any Riemannian metric g,*

$$\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2 c_1^2[M]}. \tag{1.6}$$

Moreover, the equality holds if and only if g is a Kähler-Einstein metric with negative scalar curvature.

By [Ta] the canonical class of a symplectic manifold is a monopole class. Thus, Theorem 1.1 applies to a Kähler minimal surface of general type, since by [BHPV] $K_X^2 > 0$ if X is a minimal surface of general type. We remark that Theorem 1.1 implies that $\overline{\lambda}_M$ *is not a topological invariant of the underlying manifold.* Indeed, for any pair of positive integers (m, n), so that $\frac{n}{m} \in (\frac{1}{5}, 2)$, by [BHPV] VII Theorem 8.3 there is a simply connected minimal surface X of general type so that $m = c_2(X), n = c_1^2(X)$. Let M be the blow up of X at one point. Then $c_1^2[M] = n - 1$. By Theorem 1.1 we know that $\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2(n-1)}$. On the other hand, since M is a simply connected 4-manifold of odd intersection type, by Freedman's classification it is homeomorphic to the connected sums $k\mathbb{C}P^2 \# l\overline{\mathbb{C}P^2}$ for some positive integers k, l. Since the latter admits a metric with positive scalar curvature, $\overline{\lambda}_{k\mathbb{C}P^2\#l\overline{\mathbb{C}P^2}} > 0$. This shows that $\overline{\lambda}_M$ is not a topological invariant.

A geometric consequence of Theorem 1.1 is the following comparison theorem.

Corollary 1.2. *Let (M, J, g_0) be a compact Kähler-Einstein complex surface with negative scalar curvature. If g_1 is a Riemannian metric on M with $\lambda_M(g_1) = \lambda_M(g_0)$, then*

$$Vol_{g_1}(M) \geq Vol_{g_0}(M)$$

Moreover, the equality holds if and only if g_1 is a Kähler-Einstein metric with negative scalar curvature.

One may wonder whether (M, g_1) and $M, g_0)$ are isometric in the above theorem when the equality holds. This may not be true. Indeed, there are infinitely many families of Kähler Einstein metrics on a compact complex surface M in different isometry classes with negative scalar curvature but all the same volume and same $\lambda_M(\cdot)$.

The following corollary shows a deformation rigidity of Einstein metrics on compact complex Kähler-Einstein surface with negative scalar curvature.

Corollary 1.3. *Let (M, J, g_0) be a compact Kähler-Einstein complex surface with negative scalar curvature. If $\{g_t\}$, $t \in [-1, 1]$, is a family of Einstein metrics on M with initial metric g_0, then, for any t, g_t is a Kähler-Einstein metric with negative scalar curvature.*

The above Corollary 1.3 should be compared with Corollary D in [G], where the same conclusion was obtained when g_0 has positive scalar curvature. On the other hand, under some additional technical assumptions similar results are obtained in general dimensions in [DWW] and [Ko] along a completely different line.

For a compact symplectic 4-manifold N with first Chern class c_1, the Riemann-Roch formula implies that $c_1^2[N] = 2\chi(N) + 3\tau(N)$, where $\chi(N)$ and $\tau(N)$ are the Euler characteristic and the signature of N respectively. If N admits a Kähler-Einstein metric with negative scalar curvature, we have already known $\overline{\lambda}_N = -(32\pi^2(2\chi(N) + 3\tau(N)))^{\frac{1}{2}}$. In the next theorem, we will show the exact quantity of $\overline{\lambda}_M$ where M is obtained by blowing-up N at k points.

Theorem 1.4. *Let (N, ω) be a compact symplectic 4-manifold with $b_2^+(N) > 1$. Let $M = N\sharp k\overline{\mathbb{C}P^2}$, where $k \geq 0$. If $2\chi(N) + 3\tau(N) > 0$, then*

$$\overline{\lambda}_M \leq -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}. \tag{1.7}$$

Furthermore, the equality holds if N admits a Kähler-Einstein metric.

It is known that the Seiberg-Witten invariant of connected sums vanishes if both factors have positive b_2^+. In [Ba] [BaF], a refinement of the Seiberg-Witten invariant is defined, which may not vanish for connected sums of few factors. This may be used to improve the above theorem as follows:

Theorem 1.5. *Let (N_i, ω_i), $i = 1, \cdots, \ell$, where $\ell \leq 4$, be compact symplectic 4-manifolds satisfying that $b_1(N_i) = 0$, $b_2^+(N_i) \equiv 3(\mathrm{mod} 4)$, and $\sum_{i=1}^4 b_2^+(N_i) \equiv 4(\mathrm{mod} 8)$. Assume that $c_1^2[N_1] > 0$, and $c_1^2[N_i] \geq 0$ for all i. Let X be a compact oriented 4-manifold with $b_2^+(X) = 0$, which admits a metric of positive scalar curvature. Let $M = \sharp_{i=1}^{\ell} N_i \sharp X$. Then*

$$\overline{\lambda}_M \leq -\sqrt{32\pi^2 \sum_{i=1}^{\ell} c_1^2[N_i]} \tag{1.8}$$

Furthermore, the equality holds if $N_1, \cdots, N_\ell$ admit Kähler-Einstein metrics.

The technique developed in proving Theorem 1.4 and 1.5 has an easy corollary:

Corollary 1.6. *Let (N_i, g_i), $i = 1, \cdots, l_1$, be compact Riemannian 4m-manifolds ($m \geq 2$) with holonomy $SU(2m)$ or $Sp(m)$ or $Spin(7)$, and X_j, $j = 1, \cdots, l_2$, be simply connected compact oriented spin 4m-manifolds with vanishing $\widehat{A}$-genus, $\widehat{A}(X_i) = 0$. If $M = \sharp_{i=1}^{l_1} N_i \sharp \sharp_{j=1}^{l_2} X_j$, and $\widehat{A}(M) \neq 0$, then*

$$\overline{\lambda}_M = 0.$$

Let M be a smooth compact oriented 4-manifold. By Perelman [Pe1] a critical point of $\overline{\lambda}_M(\cdot)$ is an Einstein metric. Therefore, it is interesting to ask

Question: *Can one deform a metric g to an Einstein metric through the Ricci flow, provided $\overline{\lambda}_M(g)$ is sufficiently close to the maximum $\overline{\lambda_M}$ of the λ-functional?*

This may not have a positive answer in general, of course, e.g., for a graph 3-manifold M, by [Pe2][KL] $\overline{\lambda_M} = 0$, but M can not have any Einstein metric except M is a flat manifold.

To formulate our next result, let us consider the moduli space of metrics

$$\mathcal{M}_{(\Lambda,D)} = \{g : |K_g| < \Lambda^2, diam_g < D\},$$

where $diam_g$ is the diameter, and K_g is the sectional curvature of g.

Proposition 1.7. *Let (M, J) be a compact almost complex 4-manifold satisfying that $\chi(M) \in [\frac{3}{2}\tau(M), 3\tau(M)]$, and $\tau(M) > 0$. If the canonical Spin^c-structure $\mathfrak{c}$ induced by J is a monopole class, then there exists a constant $\varepsilon = \varepsilon(\Lambda, D) > 0$ depending only on Λ and D such that for any Riemannian metric $g \in \mathcal{M}_{(\Lambda,D)}$ on M satisfying that*

$$\overline{\lambda}_M(g) \geq -\sqrt{32\pi^2(2\chi(M) + 3\tau(M))} - \varepsilon \tag{1.9}$$

it can be deformed to a complex hyperbolic metric through the Ricci flow.

The rest of the paper is organized as follows: In §2 we recall some facts about Seiberg-Witten equations. In §3 we prove Theorem 1.1, Corollary 1.2 and Corollary 1.3. In §4 we prove Theorem 1.4, Theorem 1.5 and Corollary 1.6. In §5 we prove Proposition 1.7.

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2. Preliminaries

In this section, we recall some facts about Seiberg-Witten equations. More details can be found in [N1] and [Le2].

Let (M, g) be a compact oriented Riemannian 4-manifold with a Spin^c structure $\mathfrak{c}$. Let b_2^+ denote the dimension of the space of self-dual harmonic 2-forms in M. Let $S_{\mathfrak{c}}^{\pm}$ denote the Spin^c-bundles associated to $\mathfrak{c}$, and let L be the determinant line bundle of $\mathfrak{c}$. There is a well-defined Dirac operator

$$\mathcal{D}_A : \Gamma(S_{\mathfrak{c}}^+) \longrightarrow \Gamma(S_{\mathfrak{c}}^-)$$

Let $c : \wedge^* T^* M \longrightarrow \mathrm{End}(S^+_{\mathfrak{c}} \oplus S^-_{\mathfrak{c}})$ denote the Clifford multiplication on the Spin^c-bundles, and, for any $\phi \in \Gamma(S^{\pm})$, let

$$q(\phi) = \overline{\phi} \otimes \phi - \frac{1}{2}|\phi|^2 \mathrm{id}.$$

The Seiberg-Witten equations read

$$\begin{aligned} \mathcal{D}_A \phi &= 0 \\ c(F^+_A) &= q(\phi) \end{aligned} \tag{2.1}$$

where the unknowns are a hermitian connection A on L and a section $\phi \in \Gamma(S^+_{\mathfrak{c}})$., and F^+_A is the self-dual part of the curvature of A.

A resolution of (2.1) is called *reducible* if $\phi \equiv 0$; otherwise, it is called *irreducible*. If (ϕ, A) is a resolution of (2.1), then one calculates

$$|F^+_A| = \frac{1}{2\sqrt{2}}|\phi|^2, \tag{2.2}$$

The Bochner formula reads

$$0 = 2\triangle|\phi|^2 + 4|\nabla^A \phi|^2 + R_g|\phi|^2 + |\phi|^4, \tag{2.3}$$

where R_g is the scalar curvature of g.

The Seiberg-Witten invariant can be defined by counting the irreducible solutions of the Seiberg-Witten equations (cf. [N1] [Le2]).

Definition 2.1 (K1)**.** *Let M be a smooth compact oriented 4-manifold. An element $\alpha \in H^2(M, \mathbb{Z})/torsion$ is called a monopole class of M if and only if there exists a* Spin^c*-structure $\mathfrak{c}$ on M with the first Chern class $c_1 \equiv \alpha(\text{mod torsion})$, so that the Seiberg-Witten equations have a solution for every Riemannian metric g on M.*

Deep results have been found in Seiberg-Witten theory to detect the monopole classes. For example, if (M, ω) is a compact symplectic 4-manifold with $b^+_2 > 1$, the canonical class of (M, ω) is a monopole class (cf. [Ta], or Theorem 4.2 in [K2]).

A refinement of Seiberg-Witten invariant is defined in [Ba] [BaF], which takes values in a cohomotopy group. The remarkable fact is that this invariant is not killed off by the sort of connected sum operation. If (N_i, ω_i), $i \in \{1, 2, 3, 4\}$, are the same as in Theorem 1.5, then, by Proposition 10 in [IL], $\sum_{i=1}^{l} \pm c_1(N_i)$ is a monopole class of $\sharp_{i=1}^{l} N_i$ where $l \leq 4$.

3. Proof of Theorems 1.1

Let (M, g) be a Riemannian Spin^c-manifold of dimension n. To prove Theorem 1.1, we need the following version of Kato's inequality.

Lemma 3.1. *Let ϕ be a harmonic* Spin^c*-spinor on (M, g), i.e. $\mathcal{D}_A \phi = 0$, where $\mathcal{D}_A$ is the Dirac operator and A is a connection on the determinant line bundle. Then*

$$|\nabla|\phi||^2 \leq |\nabla^A \phi|^2 \tag{3.1}$$

at all points where ϕ is non-zero. Moreover, the equality can only occur if $\nabla^A \phi \equiv 0$.

Proof. Fix a point $p \in M$ at which $\phi(p) \neq 0$ so that $|\phi|$ is differentiable at p. If $e_1, \cdots, e_n$ is an orthonormal basis of T_pM, then $|\nabla|\phi||^2 = \sum |\frac{\partial}{\partial e_i}|\phi||^2$ and $|\nabla^A \phi|^2 = \sum |\nabla^A_{e_i}\phi|^2$. For any i,

$$|\phi||\frac{\partial}{\partial e_i}|\phi|| = \frac{1}{2}|\frac{\partial}{\partial e_i}|\phi|^2| = |\mathrm{Re}\langle \nabla^A_{e_i}\phi, \phi\rangle| \leq |\nabla^A_{e_i}\phi||\phi|,$$

$$|\frac{\partial}{\partial e_i}|\phi|| \leq |\nabla^A_{e_i}\phi|, \quad \text{and} \quad |\nabla|\phi||^2 \leq |\nabla^A\phi|^2.$$

The equality can only occur if there are real numbers α_i such that $\nabla^A_{e_i}\phi = \alpha_i\phi$. Since ϕ is a harmonic Spin^c spinor,

$$0 = \mathcal{D}_A\phi = \sum c(e_i)\nabla^A_{e_i}\phi = c(\sum \alpha_i e_i)\phi = c(w)\phi,$$

where $w = \sum \alpha_i e_i$ and c is the Clifford multiplication. Then

$$0 = -|w|^2\phi.$$

Thus $w = 0$, $\alpha_i = 0$ and $\nabla^A\phi = 0$ at p. Thus we obtain the conclusion. $\square$

For any $\varepsilon > 0$, let $|\phi|^2_\varepsilon = |\phi|^2 + \varepsilon^2$. If ϕ is harmonic, by above lemma,

$$|\nabla|\phi|_\varepsilon|^2 \leq \frac{|\phi|}{|\phi|_\varepsilon}|\nabla|\phi||^2 \leq |\nabla^A\phi|^2 \tag{3.2}$$

at points where $\phi(p) \neq 0$. Since $\{p \in M | \phi(p) \neq 0\}$ is dense in M for harmonic ϕ, we conclude that (3.2) holds everywhere in M.

Proposition 3.2. *Let (M, g) be a compact oriented Riemannian 4-manifold, and $\mathfrak{c}$ be a Spin^c-structure on M. If there is an irreducible solution (ϕ, A) to the Seiberg-Witten equations (2.1) for g and $\mathfrak{c}$, then*

$$\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2[c_1^+]^2[M]}, \tag{3.3}$$

where c_1^+ is the self-dual part of the harmonic form representing the first Chern class c_1 of $\mathfrak{c}$. When $[c_1^+] \neq 0$, equality can only occur if g is a Kähler metric with constant negative scalar curvature.

Proof. Let (ϕ, A) be an irreducible solution to the Seiberg-Witten equations. The Bochner formula implies

$$0 = \frac{1}{2}\Delta|\phi|^2 + |\nabla^A\phi|^2 + \frac{R_g}{4}|\phi|^2 + \frac{1}{4}|\phi|^4,$$

Therefore

$$\int_M (|\nabla^A\phi|^2 + \frac{R_g}{4}|\phi|^2) dvol_g = -\frac{1}{4}\int_M |\phi|^4 dvol_g.$$

By (3.2),

$$\int_M (|\nabla|\phi|_\varepsilon|^2 + \frac{R_g}{4}|\phi|^2_\varepsilon) dvol_g \leq -\frac{1}{4}\int_M |\phi|^4 dvol_g + \varepsilon^2 \int_M \frac{R_g}{4} dvol_g.$$

Since $\lambda_M(g)$ is the lowest eigenvalue of the operator $-4\triangle + R_g$, we obtain

$$\lambda_M(g)\int_M |\phi|^2_\varepsilon dvol_g \leq \int_M (4|\nabla|\phi|_\varepsilon|^2 + R_g|\phi|^2_\varepsilon) dvol_g.$$

Note that, for $\varepsilon \ll 1$, $\lambda_M(g) \leq 0$. By Schwarz inequality,

$$\begin{aligned}\lambda_M(g)\mathrm{Vol}_g(M)^{\frac{1}{2}}(\int_M |\phi|_\varepsilon^4 dvol_g)^{\frac{1}{2}} &\leq \lambda_M(g)\int_M |\phi|_\varepsilon^2 dvol_g \\ &\leq -\int_M |\phi|^4 dvol_g + \varepsilon^2 \int_M R_g dvol_g.\end{aligned}$$

Letting $\varepsilon \longrightarrow 0$, we obtain

$$\overline{\lambda}_M(g) = \lambda_M(g)\mathrm{Vol}_g(M)^{\frac{1}{2}} \leq -(\int_M |\phi|^4 dvol_g)^{\frac{1}{2}}.$$

By the second equation in the Seiberg-Witten equations, we get that

$$\overline{\lambda}_M(g) \leq -(\int_M |\phi|^4 dvol_g)^{\frac{1}{2}} = -(8\int_M |F_A^+|^2 dvol_g)^{\frac{1}{2}}.$$

Note that c_1^+ is the self-dual part of the harmonic form representing the first Chern class c_1. Clearly $F_A^+ - 2\pi c_1^+$ is L^2-orthogonal to the harmonic forms space. Thus

$$\int_M |F_A^+|^2 dvol_g \geq 4\pi^2 \int_M |c_1^+|^2 dvol_g = 4\pi^2 \int_M c_1^+ \wedge c_1^+ = 4\pi^2 [c_1^+]^2$$

$$\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2[c_1^+]^2[M]}$$

If the equality holds, all of '$\leq$' above are '$=$'. By Lemma 3.1 and the Bochner formula, $\nabla^A\phi \equiv 0$, $R_g = -|\phi|^2 = \text{const.}$, and

$$\nabla F_A^+ \equiv 0.$$

Note that F_A^+ is a non-degenerate 2-form since $\phi \neq 0$. Thus, g is a Kähler metric with parallel Kähler form $\omega = \sqrt{2}\frac{F_A^+}{|F_A^+|}$. The desired result follows. □

Proof of Theorem 1.1. From the hypothesis, for any Riemannian metric g, there is a solution (ϕ, A) of the Seiberg-Witten equations. Let c_1^+ is the self-dual part of the harmonic form representing the first Chern class c_1 of $\mathfrak{c}$. Since

$$\frac{1}{8}\int_M |\phi|^4 dvol_g = \int_M |F_A^+|^2 dvol_g \geq 4\pi^2[c_1^+]^2[M] \geq 4\pi^2 c_1^2[M] > 0,$$

the solution (ϕ, A) is irreducible. By Proposition 3.2, we have

$$\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2[c_1^+]^2[M]} \leq -\sqrt{32\pi^2 c_1^2[M]}.$$

Moreover, if $[c_1^+]^2[M] \neq 0$, equality can only occur if g is a Kähler metric with constant negative scalar curvature. If g is a metric such that the equality holds in the above formula, then g is a critical point of the functional $\overline{\lambda}_M(\cdot)$. By the claim in §2.3 of [Pe1], g is a gradient soliton, i.e, we have the following equation

$$\mathrm{Ric}(g) - cg + \nabla\nabla f = 0,$$

where c is a constant, f satisfies the equation

$$-4\triangle e^{-\frac{1}{2}f} + R_g e^{-\frac{1}{2}f} = \lambda_M(g)e^{-\frac{1}{2}f}.$$

Since $\lambda_M(g)$ is the lowest eigenvalue of the operator $-4\triangle + R_g$ where R_g is a constant, we obtain that f is a constant, and g is an Einstein metric.

Now assume that g is a Kähler-Einstein metric with negative scalar curvature. We can assume that the Ricci form $\rho = -\omega$ where ω is the Kähler form associated to g. It is well known that ρ is self-dual and is the harmonic representative of $2\pi c_1$. We have

$$(32\pi^2[c_1^+]^2[M])^{\frac{1}{2}} = (32\pi^2 c_1^2[M])^{\frac{1}{2}} = 4(\frac{1}{2}\int_M \omega^2)^{\frac{1}{2}} = 4\mathrm{Vol}_g(M)^{\frac{1}{2}}.$$

Since $\lambda_M(g)$ is the lowest eigenvalue of the operator $-4\triangle - 4$, $\lambda_M(g) = -4$. Thus

$$\overline{\lambda}_M(g) = -4\mathrm{Vol}_g(M)^{\frac{1}{2}} = -\sqrt{32\pi^2 c_1^2[M]}.$$

The desired result follows. $\square$

Proofs of Corollary 1.2 and Corollary 1.3. From the hypothesis, (M, J) is a complex surface of general type with

$$c_1^2[M] = 2\chi(M) + 3\tau(M) > 0,$$

(cf. Corollary 3.5 in [Le2]). By Theorem 4.1 in [Le2], the (mod 2) Seiberg-Witten invariant $n_{\mathfrak{c}}(M) \neq 0$ where $\mathfrak{c}$ is the canonical Spin^c structure induced by J.

If g_1 is a Riemannian metric on M, then, by Theorem 1.1,

$$\lambda_M(g_1)\mathrm{Vol}_{g_1}(M)^{\frac{1}{2}} = \overline{\lambda}_M(g_1) \leq -\sqrt{32\pi^2 c_1^2[M]} = \lambda_M(g_0)\mathrm{Vol}_{g_0}(M)^{\frac{1}{2}}.$$

Thus, if $\lambda_M(g_1) = \lambda_M(g_0)$, we obtain

$$\mathrm{Vol}_{g_1}(M) \geq \mathrm{Vol}_{g_0}(M)$$

with equality if and only if g_1 is a Kähler-Einstein metric with negative scalar curvature. This proves Corollary 1.2.

To prove Corollary 1.3, let $\{g_t\}$, $t \in [0, 1]$, be a family of Einstein metrics starting at g_0 on M, i.e. $\mathrm{Ric}(g_t) = \frac{R_{g_t}}{4} g_t$. Let $f_t \in C^\infty(M)$ such that, for any t, $e^{-\frac{f_t}{2}}$ is the eigenfunction of the lowest eigenvalue of the operator $-4\triangle + R_{g_t}$ normalized by $\int_M e^{-f_t} dvol_{g_t} = 1$. Note that $\lambda_M(g_t) = R_{g_t}$, and f_t is a constant function for any $t \in [0, 1]$. For a $t_0 \in [0, 1]$, if $v_{ij} = \frac{d}{dt} g_{t,ij}|_{t=t_0}$, $h = \frac{d}{dt} f_t|_{t=t_0}$, then we have $\frac{d}{dt} dvol_{g_t}|_{t=t_0} = \frac{1}{2} v dvol_{g_{t_0}}$ where $v = g_{t_0}^{ij} v_{ij}$, and

$$\int_M e^{-f_{t_0}}(\frac{1}{2}v - h) dvol_{g_{t_0}} = 0.$$

By the first formula in Section 1 of [Pe1],

$$\begin{aligned}
\frac{d}{dt}\lambda_M(g_t)|_{t=t_0} &= \int_M e^{-f_{t_0}}[-v_{ij}\mathrm{Ric}_{t_0,ij} + (\frac{1}{2}v - h)R_{g_t}] dvol_{g_{t_0}} \\
&= -\int_M e^{-f_{t_0}} v_{ij}\mathrm{Ric}_{t_0,ij} dvol_{g_{t_0}} \\
&= -\int_M e^{-f_{t_0}} v \frac{R_{g_{t_0}}}{4} dvol_{g_{t_0}}.
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\overline{\lambda}_M(g_t)|_{t=t_0} &= \frac{d}{dt}\lambda_M(g_t)\mathrm{Vol}_{g_{t_0}}(M)^{\frac{1}{2}} + \frac{1}{2}\lambda_M(g_{t_0})\mathrm{Vol}_{g_{t_0}}(M)^{-\frac{1}{2}}\frac{d}{dt}\mathrm{Vol}_{g_t}(M)|_{t=t_0} \\
&= -\mathrm{Vol}_{g_{t_0}}(M)^{-\frac{1}{2}}\frac{R_{g_{t_0}}}{4}\int_M v dvol_{g_{t_0}} \\
&\quad +\frac{1}{4}\lambda_M(g_{t_0})\mathrm{Vol}_{g_{t_0}}(M)^{-\frac{1}{2}}\int_M v dvol_{g_{t_0}} \\
&= 0.
\end{aligned}$$

Hence

$$\frac{d}{dt}\overline{\lambda}_M(g_t) \equiv 0.$$

and so

$$\overline{\lambda}_M(g_t) \equiv \overline{\lambda}_M(g_0) = -\sqrt{32\pi^2 c_1^2[M]},$$

for any t. Therefore, by Theorem 1.1 g_t is a Kähler-Einstein metric with negative scalar curvature. Corollary 1.3 follows. □

4. Proofs of Theorem 1.4 and 1.5

To prove Theorem 1.4 and Theorem 1.5 we need the following proposition.

Proposition 4.1. *Let N and X be two smooth compact oriented n-manifolds, $n \geq 3$, and M be the connected sum of N and X, i.e. $M = N\sharp X$.*

(i) *If X admits a metric with positive scalar curvature, then*

$$\overline{\lambda}_N \leq \overline{\lambda}_M. \tag{4.1}$$

(ii) *If $\overline{\lambda}_N \leq 0$, $\overline{\lambda}_X \leq 0$, $\overline{\lambda}_M \leq 0$, and $n = 4m$, then*

$$-(\overline{\lambda}_X^{2m} + \overline{\lambda}_N^{2m})^{\frac{1}{2m}} \leq \overline{\lambda}_M. \tag{4.2}$$

We remark that the inequality above is often a strict inequality, e.g. if N is a simply connected Spin-manifold of dimension $4m \geq 5$ with $\hat{A}$-genus nonzero and $X = \mathbb{C}P^{2m}$, clearly $\overline{\lambda}_N \leq 0$, however, by [GL][SY] it is well-known that $N\#\mathbb{C}P^{2m}$ admits a metric with positive scalar curvature, therefore $\overline{\lambda}_{N\#\mathbb{C}P^{2m}} > 0$.

Lemma 4.2. *Let (X, h) be an oriented compact Riemannian n-manifold with positive scalar curvature, N be an oriented smooth compact n-manifold, $n \geq 3$, and $M = N\sharp X$. Then, for any metric g on N and $0 < \varepsilon \ll 1$, there exists a metric g_ε on M such that*

$$\lambda_N(g) - \varepsilon \leq \lambda_M(g_\varepsilon), \quad \text{and} \quad |Vol_g(N) - Vol_{g_\varepsilon}(M)| \leq \varepsilon.$$

Remark. The fact that M admits a metric g_ε such that $\lambda_M(g_\varepsilon)$ is close to $\lambda_N(g)$ is an easy consequence of Theorem 3.1 in [BD2]. Here we must construct g_ε carefully such that $\mathrm{Vol}_{g_\varepsilon}(M)$ is close to $\mathrm{Vol}_g(N)$.

Proof. For a $p \in N$, denote $U(r) = \{x | dist_g(x,p) < r\}$. By Lemma 3.7 in [BD2], there exists a $0 < \overline{r} < 1$ such that, for any $0 < r < \frac{\overline{r}^{11}}{2}$ and any smooth function u on $A(r, (2r)^{\frac{1}{11}})$, the following holds

$$\|u\|^2_{L^2(A(r,2r))} \leq 10 r^{\frac{5}{2}} \|u\|^2_{L^2(A(r,(2r)^{\frac{1}{11}}))} \tag{4.3}$$

if $\int_{\partial U(\rho)} u \partial_\nu u dA \geq 0$ holds for all $\rho \in [r, (2r)^{\frac{1}{11}}]$. Here $A(r, (2r)^{\frac{1}{11}}) = \{x | r \leq dist_g(x,p) \leq (2r)^{\frac{1}{11}}\}$, and ν is the unite normal vector field of $\partial U(\rho)$ pointing away from p. Let Λ be a positive constant bigger than the lowest eigenvalue of the operator $-4\triangle + R_g$ on $(N \backslash U(\overline{r}), g)$ with Dirichlet boundary conditions. Let R_0 be a lower bound of the scalar curvature R_g of (N, g), and R_1 be a number such that

$$R_1 > \min\{0, \Lambda\}, \qquad \Lambda \frac{\Lambda - R_0}{R_1 - \Lambda} \leq \frac{\varepsilon}{2}. \tag{4.4}$$

By the arguments in the proof of Theorem 3.1 in [BD2] or Proposition 2.1 of [BD1], there is a metric g' on N arbitrarily close to g in the C^1-topology such that $R_{g'} \geq R_0$ and $R_{g'} \geq 2R_1$ on a neighborhood U_0 of p. Since both $\lambda_N(g)$ and $Vol_g(N)$ depend continuously on g in the C^1-topology (See Lemma 3.4 in [BD2]), we may without loss of generality assume that $R_g \geq R_0$ and $R_g \geq 2R_1$ on a neighborhood U_0 of p.

Now we choose $r > 0$ and $\zeta > 0$ so small that

(i)

$$\frac{R_1 - R_0}{R_1 - \Lambda}((\Lambda + 1 - R_0)\zeta + \zeta^2) \leq \frac{\varepsilon}{2}, \tag{4.5}$$

(ii) $2\sqrt{10} r^{\frac{1}{4}} < \zeta$,
(iii) $U((2r)^{\frac{1}{11}}) \subset U_0$,
(iv) $(2r)^{\frac{1}{11}} \leq \overline{r}$,
(v) $\mathrm{Vol}_g(U(r)) < \frac{\varepsilon}{2}$.

Let η be a smooth cut-off function such that

(i) $0 \leq \eta \leq 1$ on N,
(ii) $\eta \equiv 0$ on $U(r)$,
(iii) $\eta \equiv 1$ on $N \backslash U(2r)$,
(iv) $|d\eta| \leq \frac{2}{r}$ on N.

Lemma 4.3. *For any $0 < \theta_0 \ll 1$, there is a metric $\widetilde{g}_{\theta_0}$ on $A(r, \frac{r}{2}) = U(r) \backslash U(\frac{r}{2})$ satisfying that $R_{\widetilde{g}_{\theta_0}} \geq R_1$, $\widetilde{g}_{\theta_0}$ agrees with g near the boundary $\partial U(r)$, and $\widetilde{g}_{\theta_0}$ agrees with $dt^2 + \delta^2 g_{0,1}$ near the boundary $\partial U(\frac{r}{2}) \simeq S^{n-1}(1)$, where $\delta = \delta(\theta_0)$ is a function of θ_0 such that $\delta \ll \theta_0$, and $g_{0,1}$ is the standard metric of sectional curvature 1 on $S^{n-1}(1)$. Furthermore,*

$$|Vol_{\widetilde{g}_{\theta_0}}(A(r, \frac{r}{2})) - Vol_g(U(r))| \longrightarrow 0,$$

if $\theta_0 \longrightarrow 0$.

Proof. We will use Gromov-Lawson's construction here (See Theorem A of [GL], and Theorem 3.1 of [RS]). The key idea of the proof of Theorem A in [GL] is to choose a suitable curve γ in the t-ϱ plane, and to consider

$$T_\gamma = \{(t,x) \in \mathbb{R} \times U(r) | (t, dist_g(x,p) = \varrho) \in \gamma\},$$

with the induced metric, where $\mathbb{R}$ is given the Euclidean metric and $\mathbb{R} \times U(r)$ is given the natural product metric $dt^2 + g$. The scalar curvature is given by

$$\begin{aligned} R_\gamma &= R_g + ((n-1)(n-2)\frac{1}{\varrho^2} + O(1))\sin^2\theta - (n-1)(\frac{1}{\varrho} + O(\varrho))k\sin\theta \\ &\geq 2R_1 + ((n-1)(n-2)\frac{1}{\varrho^2} - C)\sin^2\theta - (n-1)(\frac{1}{\varrho} + C'\varrho)k\sin\theta, \end{aligned}$$

where C C' are constants depending only on the curvature of g, k is the curvature of γ, and θ is the angle between γ and the ϱ-axis (See the formula (1) in [GL]). There are several steps to construct γ.

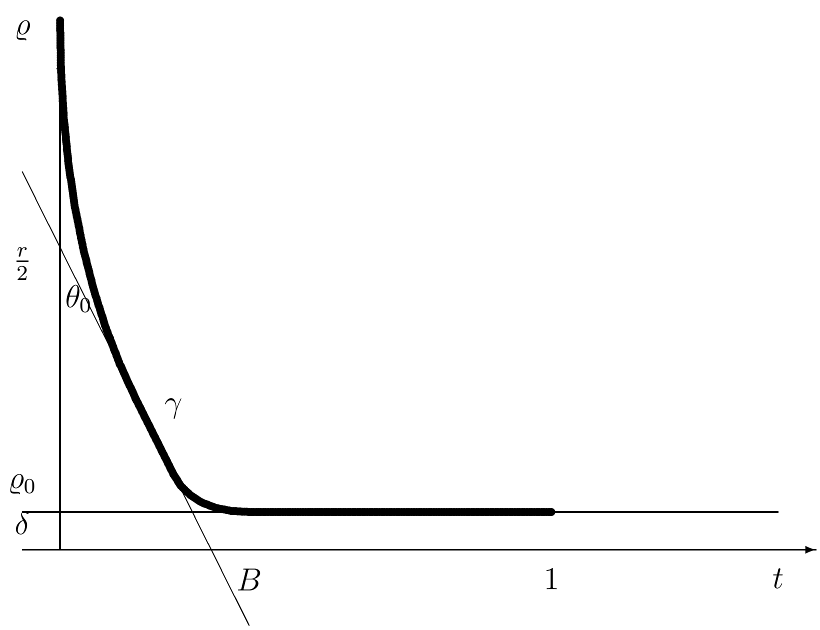

FIGURE 1.

First, let γ_0 be the bent line segment given by $\{(t,\varrho)|\varrho = -\coth\theta_0 t + \frac{r}{2}\}$ on $\mathbb{R}\times[\frac{r}{4}, 0]$, and a smooth cure with angle between γ_0 and the ϱ-axis less than θ_0 on $\mathbb{R}\times[r, \frac{r}{4}]$. From the proof of Theorem A in [GL] or the proof of Theorem 3.1 in [RS], we can choose $0 < \theta_0 \ll 1$ such that $R_{\gamma_0} \geq R_1$.

Following the arguments in P359 of [RS], we choose a ϱ_0 with $0 < \varrho_0 < \min(\sqrt{\frac{1}{4C}}, \sqrt{\frac{1}{2C'}})$. Then, for $0 < \varrho \leq \varrho_0$, we have

$$R_\gamma \geq 2R_1 + (n-1)\frac{3}{4\varrho^2}\sin^2\theta - (n-1)\frac{3}{2\varrho}k\sin\theta.$$

Let γ be γ_0 on $\mathbb{R} \times [r, \varrho_0]$, and be a curve satisfying $k = \frac{\sin\theta}{2\varrho}$ on $\mathbb{R} \times [\varrho_0, 0]$. By the arguments in P359 of [RS], γ is given by the graph of function $\varrho = f(t)$ with

$f(t) = \delta + \frac{1}{4\delta}(t-B)^2$. Note that $(\varrho_0, t_0) \in \gamma$, where $t_0 = (\frac{r}{2} - \varrho_0)\tan\theta_0$, and

$$\varrho_0 = \delta + \frac{1}{4\delta}(t_0 - B)^2, \quad f'(t_0) = \frac{1}{2\delta}(t_0 - B) = \frac{\varrho_0 - \frac{r}{2}}{t_0}.$$

Thus we have

$$\delta = \frac{t_0^2 \varrho_0}{(\varrho_0 - \frac{r}{2})^2 + t_0^2}, \quad \text{and} \quad B = t_0 + \sqrt{4\delta(\varrho_0 - \delta)}. \tag{4.6}$$

By taking $\theta_0 \ll r$ and $\varrho_0 \ll r$, we obtain

$$\delta < 4\theta_0^2 \varrho_0, \quad \text{and} \quad B < r\theta_0 + 4\theta_0\varrho_0. \tag{4.7}$$

After γ reach (B, δ), let γ be $[B, 2B] \times \{\delta\}$. Now we have constructed a metric on T_γ, denoted by g_γ, satisfying that $R_\gamma \geq R_1$, g_γ agrees with g near $\partial U(r)$, g_γ agrees with the product metric induced by $\mathbb{R} \times U(r)$ near the other boundary of T_γ, $\{2B\} \times \partial U(\delta)$. Furthermore, if we let $\theta_0 \longrightarrow 0$, then, by (4.7),

$$|\mathrm{Vol}_{g_\gamma}(T_\gamma) - \mathrm{Vol}_g(U(r))| \longrightarrow 0. \tag{4.8}$$

Note that $\partial U(\delta) \cong S^{n-1}(\delta) = \{y \in \mathbb{R}^n | \|y\| = \delta\}$. If $g_{0,1}$ is the standard metric of sectional curvature 1 on $S^{n-1}(1)$, then $\frac{1}{\delta^2} g|_{\partial U(\delta)}$ converges to $g_{0,1}$ in the C^2-topology by Lemma 1 in [GL], i.e. there is a 2-tensor $\alpha(\delta)$ on $S^{n-1}(1)$ with $\frac{1}{\delta^2} g|_{\partial U(\delta)} - g_{0,1} = \alpha(\delta)$ and $\|\alpha(\delta)\|_{C^2} \longrightarrow 0$ when $\delta \longrightarrow 0$. Let $\sigma(t)$ be a smooth function such that $\sigma(t) \equiv 1$ on $[0, \frac{1}{3}]$, $\sigma(t) \equiv 0$ on $[\frac{2}{3}, 1]$, and $|\frac{d}{dt}\sigma(t)| \leq 4$ on $[0, 1]$. Define a metric on $[0, \frac{1}{\delta}] \times S^{n-1}(1)$ by $g'_\delta = dt^2 + g_{0,1} + \sigma(\delta t)\alpha(\delta)$. When $\delta \ll 1$, $R_{g'_\delta} > \frac{1}{4}$. Define $g_\delta = \delta^2 g'_\delta$ on $[2B, 1] \times \partial U(\delta)$ which satisfies that $g_\delta = dt^2 + g|_{\partial U(\delta)}$ near $\{2B\} \times \partial U(\delta)$, $g_\delta = dt^2 + \delta^2 g_{0,1}$ near $\{1\} \times \partial U(\delta)$, $R_{g_\delta} > \frac{1}{4\delta^2}$, and $\mathrm{Vol}_{g_\delta}([2B, 1] \times \partial U(\delta)) = O(\delta^{n-1}) = O(\theta_0^{2n-2})$. Let $\widetilde{T}_\gamma$ be the manifold obtained by gluing T_γ and $[2B, 1] \times \partial U(\delta)$ at $\{2B\} \times \partial U(\delta)$, i.e.

$$\widetilde{T}_\gamma = T_\gamma \bigcup [2B, 1] \times \partial U(\delta), \tag{4.9}$$

and $\widetilde{g}_\gamma$ be a metric on $\widetilde{T}_\gamma$ such that $\widetilde{g}_\gamma = g_\gamma$ on T_γ, and $\widetilde{g}_\gamma = g_\delta$ on $[2B, 1] \times \partial U(\delta)$. Thus the metric $\widetilde{g}_\gamma$ satisfies that $R_{\widetilde{g}_\gamma} \geq R_1$ and

$$|\mathrm{Vol}_{\widetilde{g}_\gamma}(\widetilde{T}_\gamma) - \mathrm{Vol}_g(U(r))| \longrightarrow 0, \tag{4.10}$$

when $\theta_0 \longrightarrow 0$. Since $A(r, \frac{r}{2}) \simeq \widetilde{T}_\gamma$, we obtain the conclusion by letting $\widetilde{g}_{\theta_0} = \widetilde{g}_\gamma$. □

Let's continue to prove Lemma 4.2. Let $\widetilde{U}$ be the connected sum of $U(r)$ and X. Now let's consider (X, h). By the proof of Theorem A in [GL], we have a compact manifold $\widetilde{X}$ with boundary $\partial\widetilde{X} = S^{n-1}(\varsigma)$, which is obtained by deleting a small disc from X, and a metric $\widetilde{h}$ on $\widetilde{X}$ such that the scalar curvature $R_{\widetilde{h}}$ is positive, and $\widetilde{h} = dt^2 + g_{0,\varsigma}$ near the boundary $\partial\widetilde{X}$, where $g_{0,\varsigma}$ is the standard metric of sectional curvature $\frac{1}{\varsigma^2}$ on $S^{n-1}(\varsigma)$. By letting $\theta_0 \ll \min\{\varsigma, \min R_{\widetilde{h}}\}$, we obtain that $\delta \ll \min\{\varsigma, \min R_{\widetilde{h}}\}$, and the metric $(\frac{\delta}{\varsigma})^2\widetilde{h}$ satisfies that the scalar curvature of $(\frac{\delta}{\varsigma})^2\widetilde{h}$ is bigger than R_1, $(\frac{\delta}{\varsigma})^2\widetilde{h} = dt^2 + g_{0,\delta}$ near the boundary $\partial\widetilde{X}$, and

$$\mathrm{Vol}_{(\frac{\delta}{\varsigma})^2\widetilde{h}}(\widetilde{X}) \longrightarrow 0,$$

if $\delta \longrightarrow 0$.

Note that $\widetilde{U}$ is obtained by gluing $A(r,\frac{r}{2})$ and $\widetilde{X}$ at $\partial U(\frac{r}{2}) \cong \partial\widetilde{X}$, i.e. $\widetilde{U} = A(r,\frac{r}{2})\bigcup\widetilde{X}$. For any $\theta_0 \ll 1$, define a metric $\widetilde{g}'_{\theta_0}$ on $\widetilde{U}$ such that $\widetilde{g}'_{\theta_0} = \widetilde{g}_{\theta_0}$ on $A(r,\frac{r}{2})$, where $\widetilde{g}_{\theta_0}$ is the metric obtained in Lemma 4.3, and $\widetilde{g}'_{\theta_0} = (\frac{\delta}{\varsigma})^2\widetilde{h}$ on $\widetilde{X}$, which satisfies that $R_{\widetilde{g}'_{\theta_0}} \geq R_1$ and

$$|\mathrm{Vol}_{\widetilde{g}'_{\theta_0}}(\widetilde{U}) - \mathrm{Vol}_g(U(r))| < |\mathrm{Vol}_{\widetilde{g}_{\theta_0}}(A(r,\frac{r}{2})) - \mathrm{Vol}_g(U(r))| + \mathrm{Vol}_{(\frac{\delta}{\varsigma})^2\widetilde{h}}(\widetilde{X}) \longrightarrow 0,$$

when $\theta_0 \longrightarrow 0$.

Note that M is obtained by gluing $N\backslash U(r)$ and $\widetilde{U}$ at $\partial U(r)$, i.e.

$$M = (N\backslash U(r))\bigcup\widetilde{U}.$$

Define metrics g_{θ_0} on M by $g_{\theta_0} = g$ on $N\backslash U(r)$ and $g_{\theta_0} = \widetilde{g}'_{\theta_0}$ on $\widetilde{U}$, which satisfy

$$|\mathrm{Vol}_g(N) - \mathrm{Vol}_{g_{\theta_0}}(M)| \leq \mathrm{Vol}_g(U(r)) + |\mathrm{Vol}_g(U(r)) - \mathrm{Vol}_{\widetilde{g}_{\theta_0}}(\widetilde{U})| \longrightarrow 0,$$

when $\theta_0 \longrightarrow 0$. Thus, for any $0 < \varepsilon \ll 1$, there is a θ_0 such that

$$|\mathrm{Vol}_g(N) - \mathrm{Vol}_{g_{\theta_0}}(M)| < \varepsilon. \tag{4.11}$$

By defining $g_\varepsilon = g_{\theta_0}$ on M, we obtain the volumes inequality.

Lemma 4.4.

$$\lambda_N(g) - \varepsilon \leq \lambda_M(g_\varepsilon).$$

Proof. The following arguments is the same as the proof of Theorem 3.1 in [BD2]. But for reader's convenience, we present the proof here. Let u be the eigenfunction of $\lambda_M(g_\varepsilon)$ on (M, g_ε). The function $v = \eta u$ can be regarded as a function on (N, g). Thus

$$\lambda_N(g) \leq \frac{\int_N (4|dv|^2 + R_g v^2) dvol_g}{\int_N v^2 dvol_g}. \tag{4.12}$$

Since Λ is larger than the lowest eigenvalue of the operator $-4\triangle + R_g$ on $(N\backslash U(\overline{r}), g)$ with Dirichlet boundary conditions, we have $\lambda_M(g_\varepsilon) \leq \Lambda$ by Lemma 92.5 in [KL]. Thus

$$\begin{aligned} R_1\int_{\widetilde{U}\bigcup A(r,2r)} u^2 dvol_{g_\varepsilon} + R_0\int_{M\backslash\widetilde{U}\bigcup A(r,2r)} u^2 dvol_{g_\varepsilon} &\leq \int_M (4|du|^2 + R_{g_\varepsilon}u^2) dvol_{g_\varepsilon} \\ &\leq \Lambda\int_M u^2 dvol_{g_\varepsilon}. \end{aligned}$$

Hence

$$\int_{\widetilde{U}\bigcup A(r,2r)} u^2 dvol_{g_\varepsilon} \leq \frac{\Lambda - R_0}{R_1 - R_0}\int_M u^2 dvol_{g_\varepsilon}. \tag{4.13}$$

We have

(4.14)

$$\begin{aligned}\|v\|^2_{L^2(N)} &= \|\eta u\|^2_{L^2(N)} \geq \|u\|^2_{L^2(N\setminus U(2r))}\\ &\geq (1-\frac{\Lambda - R_0}{R_1 - R_0})\|u\|^2_{L^2(M)}\\ &\geq \frac{R_1-\Lambda}{R_1-R_0}\|u\|^2_{L^2(M)}.\end{aligned}$$

For a $\rho \in [r,(2r)^{\frac{1}{11}}]$, set $\widehat{U}_\rho = \widetilde{U}\bigcup A(r,\rho)$. Since $R_{g_\varepsilon} \geq R_1$ on $\widehat{U}_\rho$, we have

$$\begin{aligned}\lambda_M(g_\varepsilon)\|u\|^2_{L^2(\widehat{U}_\rho)} &= 4\int_{\widehat{U}_\rho}\langle \triangle u, u\rangle dvol_{g_\varepsilon} + \int_{\widehat{U}_\rho} R_{g_\varepsilon}u^2 dvol_{g_\varepsilon}\\ &= 4\int_{\widehat{U}_\rho}|du|^2 dvol_{g_\varepsilon} - 4\int_{\partial\widehat{U}_\rho} u\partial_\nu u dA + \int_{\widehat{U}_\rho} R_{g_\varepsilon}u^2 dvol_{g_\varepsilon}\\ &\geq -4\int_{\partial\widehat{U}_\rho} u\partial_\nu u dA + R_1\|u\|^2_{L^2(\widehat{U}_\rho)}.\end{aligned}$$

Hence

$$4\int_{\partial\widehat{U}_\rho} u\partial_\nu u dA \geq (R_1 - \lambda_M(g_\varepsilon))\|u\|^2_{L^2(\widehat{U}_\rho)} \geq (R_1-\Lambda)\|u\|^2_{L^2(\widehat{U}_\rho)} \geq 0.$$

By Lemma 3.7 in [BD2],

$$\|u\|^2_{L^2(A(r,2r))} \leq 10r^{\frac{5}{2}}\|u\|^2_{L^2(A(r,(2r)^{\frac{1}{11}}))} \leq 10r^{\frac{5}{2}}\|u\|^2_{L^2(M)}.$$

Thus

$$\frac{2}{r}\|u\|_{L^2(A(r,2r))} \leq \zeta\|u\|_{L^2(M)}. \tag{4.15}$$

We have

$$\begin{aligned}\|dv\|^2_{L^2(N)} &= \|d(\eta u)\|^2_{L^2(N)}\\ &\leq (\|\eta du\|_{L^2(N)} + \|d\eta u\|_{L^2(N)})^2\\ &\leq (\|du\|_{L^2(M)} + \frac{2}{r}\|u\|_{L^2(A(r,2r))})^2\\ &\leq (\|du\|_{L^2(M)} + \zeta\|u\|_{L^2(M)})^2\\ &\leq (1+\zeta)\|du\|^2_{L^2(M)} + \zeta(1+\zeta)\|u\|^2_{L^2(M)}\end{aligned}$$

Since $R_{g_\varepsilon} > R_1 > 0$ on $\widetilde{U}\bigcup U(2r)$,

$$(R_g v, v)_{L^2(N)} \leq (R_{g_\varepsilon}\eta u, \eta u)_{L^2(M)} \leq (R_{g_\varepsilon}u,u)_{L^2(M)}. \tag{4.16}$$

We obtain

$$\begin{aligned}&4\|dv\|^2_{L^2(N)} + (R_g v, v)_{L^2(N)}\\ \leq\ & 4(1+\zeta)\|du\|^2_{L^2(M)} + \zeta(1+\zeta)\|u\|^2_{L^2(M)} + (R_{g_\varepsilon}u,u)_{L^2(M)}\\ \leq\ & (1+\zeta)\lambda_M(g_\varepsilon)\|u\|^2_{L^2(M)} - \zeta R_0\|u\|^2_{L^2(M)} + \zeta(1+\zeta)\|u\|^2_{L^2(M)}\\ \leq\ & [\lambda_M(g_\varepsilon) + (\Lambda+1-R_0)\zeta + \zeta^2]\|u\|^2_{L^2(M)}.\end{aligned}$$

Hence we have

$$\begin{aligned}\lambda_N(g) &\leq \frac{\int_N (4|dv|^2 + R_g v^2) dvol_g}{\int_N v^2 dvol_g} \\ &\leq \frac{R_1 - R_0}{R_1 - \Lambda}[\lambda_M(g_\varepsilon) + (\Lambda + 1 - R_0)\zeta + \zeta^2] \\ &= \lambda_M(g_\varepsilon) + \frac{\Lambda - R_0}{R_1 - \Lambda}\lambda_M(g_\varepsilon) + \frac{R_1 - R_0}{R_1 - \Lambda}[(\Lambda + 1 - R_0)\zeta + \zeta^2] \\ &\leq \lambda_M(g_\varepsilon) + \varepsilon,\end{aligned}$$

by (4.4), (4.5) and (4.14). Thus both Lemma 4.4 and Lemma 4.2 are proved. □

□

Lemma 4.5. *Let N_1 and N_2 be two compact oriented 4m-manifolds with $\overline{\lambda}_{N_1} \leq 0$, $\overline{\lambda}_{N_2} \leq 0$. Let $M = N_1 \sharp N_2$. Assume that $\overline{\lambda}_M \leq 0$. For any metrics g_1 and g_2 on N_1 and N_2 respectively with $\lambda_{N_1}(g_1) = \lambda_{N_2}(g_2) = -1$, and $0 < \varepsilon \ll 1$, there is a metric g_ε on M such that*

$$(1+\varepsilon)^{2m}(\overline{\lambda}_{N_1}(g_1)^{2m} + \overline{\lambda}_{N_2}(g_2)^{2m} + \varepsilon) \geq \overline{\lambda}_M(g_\varepsilon)^{2m}.$$

Proof. Let $N = N_1 \bigcup N_2$, and $p_i \in N_i$, $i = 1, 2$. Notations, r, $\overline{r}$, $U(r)$, R_0, R_1, Λ, and ζ, are the same as in the proof of Lemma 4.2. Here the only difference is that we use the set $\{p_1, p_2\}$ in stead of a point of N. Denote $U_i(r) = \{x \in N_i | \mathrm{dist}_{g_i}(x, p_i) \leq r\}$. By Lemma 4.3, for any $0 < \theta_0 \ll 1$, for each i, there is a metric $\widetilde{g}_{i,\theta_0}$ on $A_i(r, \frac{r}{2}) = U_i(r) \backslash U_i(\frac{r}{2})$ satisfying that $R_{\widetilde{g}_{i,\theta_0}} \geq R_1$, $\widetilde{g}_{i,\theta_0}$ agrees with g_i near the boundary $\partial U_i(r)$, and $\widetilde{g}_{i,\theta_0}$ agrees with $dt^2 + \delta(\theta_0)^2 g_{0,1}$ near the boundary $\partial U_i(\frac{r}{2}) \simeq S^{n-1}(1)$, where $g_{0,1}$ is the standard metric of sectional curvature 1 on $S^{n-1}(1)$. From (4.6), we can choose $\delta = \delta(\theta_0)$ as a function of θ_0 in-dependent of i such that $\delta \ll \theta_0$. Furthermore,

$$|\mathrm{Vol}_{\widetilde{g}_{i,\theta_0}}(A_i(r, \frac{r}{2})) - \mathrm{Vol}_{g_i}(U_i(r))| \longrightarrow 0,$$

if $\theta_0 \longrightarrow 0$. Note that M is obtained by gluing $N_1 \backslash U_1(\frac{r}{2})$ and $N_2 \backslash U_2(\frac{r}{2})$ at $U_1(\frac{r}{2}) \simeq U_2(\frac{r}{2})$, i.e. $M = N_1 \backslash U_1(\frac{r}{2}) \bigcup N_2 \backslash U_2(\frac{r}{2})$. Define a metric g_{θ_0} on M by $g_{\theta_0} = g_i$ on $N_i \backslash U_i(r)$, and $g_{\theta_0} = \widetilde{g}_{i,\theta_0}$ on $A_i(r, \frac{r}{2})$, which satisfies

$$|\mathrm{Vol}_{g_{\theta_0}}(M) - \sum \mathrm{Vol}_{g_i}(N_i)| \leq \sum |\mathrm{Vol}_{\widetilde{g}_{i,\theta_0}}(A_i(r, \frac{r}{2})) - \mathrm{Vol}_{g_i}(U_i(r))| \longrightarrow 0,$$

when $\theta_0 \longrightarrow 0$. For any $\varepsilon > 0$, by letting $\theta_0 \ll 1$ and $g_\varepsilon = g_{\theta_0}$, we find a metric g_ε on M with

$$|\mathrm{Vol}_{g_\varepsilon}(M) - \sum \mathrm{Vol}_{g_i}(N_i)| \leq \varepsilon.$$

By the same arguments as in the proof of Lemma 4.4, we have

$$-1 - \varepsilon \leq \lambda_M(g_\varepsilon).$$

Thus

$$(1+\varepsilon)^{2m}(\overline{\lambda}_{N_1}(g_1)^{2m} + \overline{\lambda}_{N_2}(g_2)^{2m} + \varepsilon) \geq \overline{\lambda}_M(g_\varepsilon)^{2m}.$$

□

Proof of Proposition 4.1. First, we assume that there is a metric h on X with positive scalar curvature. By Lemma 4.2, for any metric g on N and $0 < \varepsilon \ll 1$, there exists a metric g_ε on M such that

$$\lambda_N(g) - \varepsilon \leq \lambda_M(g_\varepsilon), \quad \text{and} \quad |\mathrm{Vol}_g(N) - \mathrm{Vol}_{g_\varepsilon}(M)| \leq \varepsilon.$$

Thus we have

$$(\lambda_N(g) - \varepsilon)(\mathrm{Vol}_g(N) + \varepsilon)^{\frac{2}{n}} \leq \lambda_M(g_\varepsilon)\mathrm{Vol}_{g_\varepsilon}(M)^{\frac{2}{n}} \leq \overline{\lambda}_M, \quad \text{or}$$

$$(\lambda_N(g) - \varepsilon)(\mathrm{Vol}_g(N) - \varepsilon)^{\frac{2}{n}} \leq \lambda_M(g_\varepsilon)\mathrm{Vol}_{g_\varepsilon}(M)^{\frac{2}{n}} \leq \overline{\lambda}_M.$$

By letting $\varepsilon \longrightarrow 0$,

$$\lambda_N(g)\mathrm{Vol}_g(N)^{\frac{2}{n}} \leq \overline{\lambda}_M.$$

Thus

$$\overline{\lambda}_N = \sup_{g \in \mathcal{M}} \overline{\lambda}_N(g) \leq \overline{\lambda}_M,$$

where $\mathcal{M}$ is the set of Riemannian metrics on N. Hence, we obtain (4.1).

Now we assume that $\overline{\lambda}_N \leq 0$, $\overline{\lambda}_X \leq 0$, $\overline{\lambda}_M \leq 0$, and $n = 4m$. We can choose any two metrics g_1 and g_2 on N and X respectively with $\overline{\lambda}_N(g_1) < 0$ and $\overline{\lambda}_X(g_2) < 0$. After re-scaling them, we can assume $\lambda_N(g_1) = \lambda_X(g_2) = -1$. By Lemma 4.5, for any $\varepsilon > 0$, there exists a metric g_ε on M such that

$$-(1+\varepsilon)(\overline{\lambda}_N(g_1)^{2m} + \overline{\lambda}_X(g_2)^{2m} + \varepsilon)^{\frac{1}{2m}} \leq \overline{\lambda}_M(g_\varepsilon) \leq \overline{\lambda}_M.$$

By letting $\varepsilon \longrightarrow 0$, we obtain

$$-(\overline{\lambda}_N(g_1)^{2m} + \overline{\lambda}_X(g_2)^{2m})^{\frac{1}{2m}} \leq \overline{\lambda}_M.$$

Thus

$$-(\overline{\lambda}_N^{2m} + \overline{\lambda}_X^{2m})^{\frac{1}{2m}} = -((\sup_{g_1 \in \mathcal{M}_1} \overline{\lambda}_N(g_1))^{2m} + (\sup_{g_2 \in \mathcal{M}_2} \overline{\lambda}_X(g_2))^{2m})^{\frac{1}{2m}} \leq \overline{\lambda}_M.$$

□

Proof of Theorem 1.4. By Theorem 1 in [Ta], the Spin^c structure induced by a compatible almost complex structure on (N, ω) has Seiberg-Witten invariant equal to ± 1. By Lemma 1 in Section 3 of [Le4], for any metric g on M, we can choose a Spin^c structure on M with non-vanishing Seiberg-Witten invariant, and

$$[c_1^+]^2[M] \geq 2\chi(N) + 3\tau(N) > 0.$$

Thus, by Proposition 3.2, we obtain

$$\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2[c_1^+]^2[M]} \leq -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}.$$

Hence

$$\overline{\lambda}_M \leq -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}.$$

If N admits a Kähler-Einstein metric, then, by Proposition 4.1,

$$-\sqrt{32\pi^2(2\chi(N) + 3\tau(N))} = \overline{\lambda}_N \leq \overline{\lambda}_M \leq -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}.$$

Hence we obtain the conclusion. □

Lemma 4.6. *Let M be a spin manifold with non-vanishing $\widehat{A}$-genus, i.e. $\widehat{A}(M)\neq 0$. Then*
$$\overline{\lambda}_M\leq 0.$$

Proof. Since $\widehat{A}(M)\neq 0$, for any metric g' on M, there is a non-vanishing harmonic spinor $\phi\in\Gamma(S)$ with $\int_M|\phi|^2dvol_{g'}=1$, where S is the spin bundle. The Bochner formula implies that
$$0=\mathcal{D}^2\phi=\nabla^*\nabla\phi+\frac{R_{g'}}{4}\phi,$$
where $\mathcal{D}:\Gamma(S)\longrightarrow\Gamma(S)$ is the Dirac operator. By (3.2),
$$\begin{aligned}\lambda_M(g') &\leq \int_M(4|\nabla|\phi|_\varepsilon|^2+R_{g'}|\phi|_\varepsilon^2)dvol_{g'}\\ &\leq \int_M(4|\nabla\phi|^2+R_{g'}|\phi|^2)dvol_{g'}+\varepsilon^2\int_M R_{g'}dvol_{g'}=\varepsilon^2\int_M R_{g'}dvol_{g'}.\end{aligned}$$
By letting $\varepsilon\longrightarrow 0$, we obtain that
$$\lambda_M(g')\leq 0,\quad \text{and}\quad \overline{\lambda}_M\leq 0.$$
□

Proof of Theorem 1.5. By Theorem 1 in [Ta], for any i, the Spin^c structure induced by a compatible almost complex structure on (N_i,ω_i) has Seiberg-Witten invariant equal to ± 1. By Corollary 11 in [IL], for any metric g on M, there is a monopole class α of M satisfying that
$$[\alpha^+]^2[M]\geq\sum_{i=1}^{\ell}c_1^2[N_i].$$
Thus, by Proposition 3.2, we obtain
$$\overline{\lambda}_M(g)\leq-\sqrt{32\pi^2[\alpha^+]^2[M]}\leq-\sqrt{32\pi^2\sum_{i=1}^{\ell}c_1^2[N_i]}.$$
Hence
$$\overline{\lambda}_M\leq-\sqrt{32\pi^2\sum_{i=1}^{\ell}c_1^2[N_i]}.$$

Now, we assume that, for each i, N_i admits a Kähler-Einstein metric g_i. If the scalar curvature of g_i is negative, we have already known that $\overline{\lambda}_{N_i}=\overline{\lambda}_{N_i}(g_i)=-\sqrt{32\pi^2c_1^2(N_i)}$ by Theorem 1.1. If the scalar curvature of g_i is zero, then N_i is a $K3$-surface from the hypothesis (cf. [BHPV]). By Lemma 4.6, we obtain $0=\overline{\lambda}_{N_i}(g_i)\leq\overline{\lambda}_{N_i}\leq 0$. Thus $\overline{\lambda}_{N_i}=0=-\sqrt{32\pi^2c_1^2(N_i)}$. By Proposition 4.1,
$$-\sqrt{32\pi^2\sum_{i=1}^{\ell}c_1^2[N_i]}=-\sqrt{\sum_{i=1}^{l}\overline{\lambda}_{N_i}^2}\leq\overline{\lambda}_M\leq-\sqrt{32\pi^2\sum_{i=1}^{l}c_1^2[N_i]}.$$
This proves the desired result. □

Proof of Corollary 1.6. Note that N_i are spin manifolds (See [J]), and thus is M. By Lemma 4.6, $\overline{\lambda}_M \leq 0$ as $\widehat{A}(M) \neq 0$. Since $X_1 \cdots X_l$ are simply connected compact oriented spin n-manifolds with $\widehat{A}(X_j) = 0$, $n \geq 8$, and $n = 0 \mathrm{mod} 4$, for any X_j, there is a metric h_j on X_j with positive scalar curvature from Theorem A in [St]. Note that (N_i, g_i) are Ricci-flat Einstein manifolds with $\widehat{A}(N_i) \neq 0$ (See [J]). By Lemma 4.6, $0 = \overline{\lambda}_{N_i}(g_i) \leq \overline{\lambda}_{N_i} \leq 0$. By Proposition 4.1,

$$0 \leq \overline{\lambda}_{\sharp_{i=1}^{l_1} N_i} \leq \overline{\lambda}_M \leq 0.$$

We obtain the conclusion. □

5. Proof of Proposition 1.7

Proof of Proposition 1.7. If it is not true, there exists a sequence of metrics $\{g_k\} \subset \mathcal{M}_{(\Lambda, D)}$ such that

$$-\sqrt{32\pi^2(2\chi(M) + 3\tau(M))} - \frac{1}{k} \leq \overline{\lambda}_M(g_k)$$

but g_k can never be deformed to a complex hyperbolic metric through the Ricci flow for every k.

Since $\chi(M) > 0$, there is a positive constant v independent of k such that $\mathrm{Vol}_{g_k}(M) \geq v$ by the Gauss-Bonnett-Chern theorem. By the Cheeger-Gromov theorem (cf [A]), $\{g_k\}$ has a $C^{1,\alpha}$-convergence subsequence, denoted by $\{g_k\}$ also. Therefore, there are diffeomorphisms F_k of M such that a subsequence of $\{F_k^* g_k\}$ converges, in the $C^{1,\alpha}$-topology on M, to a $C^{1,\alpha}$-metric g_∞. In fact, $\{F_k^* g_k\}$ converges in the $L^{2,p}$-topology, for any $p \geq 1$, and g_∞ is a $L^{2,p}$-metric (See [A] for details). Thus $\overline{\lambda}_M(g_\infty)$ is well defined satisfying that

$$-\sqrt{32\pi^2(2\chi(M) + 3\tau(M))} \leq \overline{\lambda}_M(g_\infty).$$

This together with Theorem 1.1 implies that

$$\overline{\lambda}_M(g_\infty) = -\sqrt{32\pi^2(2\chi(M) + 3\tau(M))},$$

and g_∞ is a Kähler-Einstein metric with negative scalar curvature. By Theorem 5 in [Le1] $\chi(M) \geq 3\tau(M)$. This together with the assumption $\chi(M) \in [\frac{3}{2}\tau(M), 3\tau(M)]$ implies that $\chi(M) = 3\tau(M)$. By Theorem 5 in [Le1] once again we know that g_∞ is a complex hyperbolic metric.

To prove the metric can be deformed to a complex hyperbolic metric through the Ricci flow, we need to smooth the $C^{1,\alpha}$-convergence to a C^2-convergence by Ricci flow. By the main theorem in [BOR] (See Theorem 5.1 in [Fu] for this version), given any $1 \gg \epsilon > 0$ and $j \in \mathbb{N}$, there exists a constant $C(j, \epsilon)$ and a smoothing operator $S_\epsilon : \mathcal{M}_{(\Lambda, D)} \longrightarrow \mathcal{M}_{(2\Lambda, 2D)}$ such that

(i) $\|S_\epsilon(g) - g\|_{C^0} < \epsilon$,
(ii) $\|\nabla^{S_\epsilon(g)} - \nabla^g\|_{C^0} < \epsilon$,
(iii) $\|\nabla^j \mathrm{Rm}(S_\epsilon(g))\|_{C^0} < C(j, \epsilon)\|\mathrm{Rm}(g)\|_{C^0}$,

where $\mathrm{Rm}(g)$ is the curvature operator of g. The proof of this result is by considering the Ricci-flow evolution equation with initial metric $g \in \mathcal{M}_{(\Lambda,D)}$

$$\begin{aligned} \tfrac{\partial}{\partial t} g(t) &= -2\mathrm{Ric}(g(t)) \\ g(0) &= g, \end{aligned}$$

and letting $S_\epsilon(g) = g(\epsilon)$. By using the operator S_ϵ to metrics g_k, we obtain a sequence of metrics $\{S_\epsilon(g_k)\} \subset \mathcal{M}_{(2\Lambda,2D)}$. Let $\widetilde{g}_k = S_\epsilon(g_k)$. By the claim in §2.3 of [Pe1], $\overline{\lambda}_M(g)$ is non-decreasing along the Ricci flow if $\overline{\lambda}_M(g) \leq 0$. Thus

$$-\sqrt{32\pi^2(2\chi(M) + 3\tau(M))} - \frac{1}{k} \leq \overline{\lambda}_M(g_k) \leq \overline{\lambda}_M(\widetilde{g}_k).$$

By the Cheeger-Gromov Theorem again, there are diffeomorphisms $\widetilde{F}_k$ of M such that a subsequence of $\{\widetilde{F}_k^* \widetilde{g}_k\}$, saying $\{\widetilde{F}_k^* \widetilde{g}_k\}$ again, which converges in the $C^{1,\alpha}$-topology in M to a $C^{1,\alpha}$-metric $\widetilde{g}_\infty$, a Kähler-Einstein metric with negative scalar curvature by Theorem 1.1. Since $\|\nabla \mathrm{Rm}(\widetilde{g}_k)\|_{C^0} < C(1,\epsilon)\Lambda$, by the Arzela-Ascoli Theorem we get a sub-sequence of $\{\mathrm{Rm}(\widetilde{F}_k^* \widetilde{g}_k)\}$ which C^0-converges to $\mathrm{Rm}(\widetilde{g}_\infty)$. Therefore, $\{\widetilde{F}_k^* \widetilde{g}_k\}$ C^2-converges to $\widetilde{g}_\infty$. As above by [Le1] $\widetilde{g}_\infty$ is a complex hyperbolic metric. Note that the sectional curvature $K(\widetilde{g}_\infty)$ of a complex hyperbolic metric is negative, i.e. there are constants μ_1 μ_2 such that $-\mu_1^2 \leq K(\widetilde{g}_\infty) \leq -\mu_2^2$. Thus, for $k \gg 1$, we have $-2\mu_1^2 \leq K(\widetilde{g}_k) \leq -\frac{1}{2}\mu_2^2$. Moreover, the Einstein tensors satisfy that

$$T_{\widetilde{g}_k} = \mathrm{Ric}(\widetilde{g}_k) - \frac{R_{\widetilde{g}_k}}{4}\widetilde{g}_k \longrightarrow 0$$

in the C^0-sense when $k \longrightarrow \infty$. By the corollary of Theorem 1.1 in [Ye], for a $k \gg 1$, $\widetilde{g}_k$ can be deformed to an Einstein metric, which is complex hyperbolic metric by [Le1] again.

Note that we first deform g_k to $\widetilde{g}_k$ through the Ricci flow, then deform $\widetilde{g}_k$ to a complex hyperbolic metric through the Ricci flow again. A contradiction. The desired result follows. □

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